Numerical analysis of an energy-like minimization method to solve Cauchy problem with noisy data

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Abstract

This paper is concerned with solving the Cauchy problem for an elliptic equation by minimizing an energy-like error functional and by taking into account noisy Cauchy data. After giving some fundamental results, numerical convergence analysis of the energy-like minimization method is carried out and leads to adapted stopping criteria for the minimization process depending on the noise rate. Numerical examples involving smooth and singular data are presented.

1 Introduction

The Cauchy problem considered here consists of solving a partial differential equation on a domain for which over-specified boundary conditions are given on a part of its boundary, which means solving a data completion problem and recover the missing boundary conditions on the remaining part of the boundary. This kind of problem arises in many industrial, engineering or biomedical applications.

Since Hadamard's works [1], the Cauchy problem is known to be ill-posed and an important numerical instability may occur during the numerical resolution of this kind of problem. It provides researchers with an interesting challenge to carry out a numerical procedure approximating the solution of the Cauchy problem in the particular case of noisy data. Many theoretical and applied works were proposed about this subject, using the Steklov-Poincaré theory (see [2, 3, 4]), regularization methods (see [5, 6]), quasi-reversibility method (see [7]) or minimal error methods (see [8, 9, 10]).

In this paper, we focus on a method introduced in [11, 12, 13] based on minimization of an energy-like functional. More precisely, in the approach proposed here, we introduce two distinct fields, each of them meeting only one of the over-specified data. They are then solutions of two well-posed problems which avoids the need of regularization methods in the case of free noise Cauchy data. Next, an energy-like error functional is introduced to measure the gap between these two fields. Then, the Cauchy problem solution is obtained when the functional reaches its minimum. This method provides hopeful results, nevertheless, as many other methods, it becomes unstable in the case of noisy data. In order to overcome this numerical instability, we propose adequate stopping criteria parametrized by the noise rate by means of numerical convergence analysis.

The outline of the paper is as follows. In section 2, we give the Cauchy problem and report classical theoretical results. In section 3, we formulate the Cauchy problem as a data completion problem and introduce the related minimization problem. In section 4 and 5, the finite element discretization, the convergence analysis and study of noise effects for the introduced minimization problem are presented respectively. We give here a priori error estimates taking into account data noise and propose stopping criteria to control instability of the minimization process. Finally, the numerical procedure and results are presented.

2 Statement of problem

We consider a Lipschitz bounded domain Ω in $\mathbb{R}^d$, $d = 2, 3$ with n the outward unit normal to the boundary $\Gamma = \partial\Omega$. Assume that Γ is partitioned into two parts Γ_u and Γ_m, of non-vanishing measure and such that $\Gamma_u \cap \Gamma_m = \emptyset$.

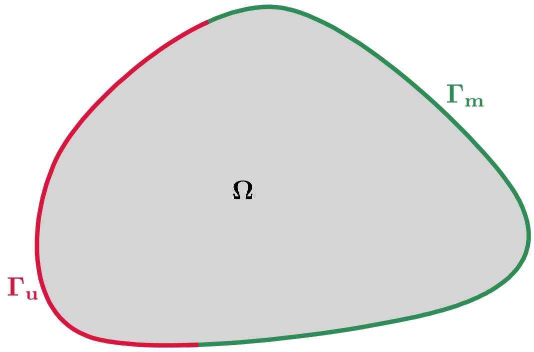


Figure 1: An example of geometry

A more common problem consists of temperature recovering in a given domain Ω assuming temperature distribution and heat flux are given over the accessible region of the boundary. Given a source term f and a contuctivity field k in Ω, a flux ϕ and the corresponding temperature T on Γ_m, we would like to recover the corresponding flux and temperature on Γ_u. The Cauchy problem is then written as :

$$\begin{cases} -\nabla \cdot \big(k(x)\nabla u\big) = f \text{ in } \Omega \\ k(x)\nabla u \cdot n = \phi \text{ on } \Gamma_m \\ u = T \text{ on } \Gamma_m \end{cases} \tag{1}$$

A problem is well-posed in the sense of Hadamard (see [1, 14, 5]) if it fulfills the three following properties : uniqueness and existence of the solution and stability. The extended Holmgren's theorem to the Sobolev spaces (see [14]) guarantees uniqueness under regularity assumptions on a solution of the Cauchy problem. As the well known Cauchy-Kowalevsky theorem (see [15]) is applicable only in the case of analytic data, the existence of this solution is then a caution to a verification of a compatibility condition which can hardly be explicitly formulated. This compatibility condition added to the fact that, for one fixed data, the set of compatible data is dense in the set of all data (see [16]), wich implies that the stability assumption is not satisfied in the sense that the dependence of the solution u of (1) on the data (ϕ, T) is not continuous. Hereafter, we assume that data (ϕ, T) are compatible.

Some notations : Let x be a generic point of Ω. The space of squared integrable functions $L^2(\Omega)$ is endowed with a natural inner product written $(\cdot,\cdot)_{L^2(\Omega)}$. The associated norm is written $\|\cdot\|_{0,\Omega}$. We note $H^1(\Omega)$ the Sobolev space of functions of $L^2(\Omega)$ for which their first order derivatives are also in $L^2(\Omega)$. Its norm and semi-norm are written $\|\cdot\|_{1,\Omega}$ and $|\cdot|_{1,\Omega}$ respectively. Let $\gamma \subset \Gamma$, we define the space $H^1_{0,\gamma}(\Omega) = \{v \in H^1(\Omega); v_{|\gamma} = 0\}$ and $H^{1/2}_{00}(\gamma)$ is the space of restrictions to γ of the functions of $H^{1/2}(\Omega) = tr\big(H^1(\Omega)\big)$. Its topological dual is written $H_{00}^{-1/2}(\gamma) = \big(H^{1/2}_{00}(\gamma)\big)'$. The associated norms are written $\|\cdot\|_{1/2,00,\gamma}$ and $\|\cdot\|_{-1/2,00,\gamma}$ respectively and $\langle\cdot,\cdot\rangle_{1/2,00,\gamma}$ states for the duality inner product.

3 Energy-like minimization method

Let $f \in L^2(\Omega)$, $k(x) \in L^\infty(\Omega)$ positive, $\phi \in H_{00}^{-1/2}(\Gamma_m)$ and $T \in H_{00}^{1/2}(\Gamma_m)$. The Cauchy problem can be written as a data completion problem :

Find $(\varphi, t) \in H_{00}^{-1/2}(\Gamma_u) \times H_{00}^{1/2}(\Gamma_u)$ such that there exists $u \in H^1(\Omega)$ solution of

$$\begin{cases} -\nabla \cdot \big(k(x)\nabla u\big) = f \text{ in } \Omega \\ u = T,\ k(x)\nabla u \cdot n = \phi \text{ on } \Gamma_m \\ u = t,\ k(x)\nabla u \cdot n = \varphi \text{ on } \Gamma_u \end{cases} \tag{2}$$

Remark 1 : We note that in the case $\bar{\Gamma}_u \cap \bar{\Gamma}_m = \emptyset$, as given in figure 2 illustrating the ring numerical tests of section 6.2, the spaces $H^{-1/2}(\Gamma_u) \times H^{1/2}(\Gamma_u)$ and $H^{-1/2}(\Gamma_m) \times H^{1/2}(\Gamma_m)$ for unknowns and data respectively would be more appropriate. Nevertheless, the general functional framework is not restrictive because the spaces $H_{00}^s(\Gamma_u)$ and $H_{00}^s(\Gamma_m)$ are dense in $H^s(\Gamma_u)$ and $H^s(\Gamma_m)$ for $s = \pm 1/2$, respectively.

The functional spaces being given, we introduce more precisely the concept of density mentioned in the previous section. We recall the following theorem :

Theorem 3.1 (i) For a fixed $T \in H_{00}^{1/2}(\Gamma_m)$, the set of data ϕ for which there exists a solution $u \in H^1(\Omega)$ to the Cauchy problem (1) is everywhere dense in $H_{00}^{-1/2}(\Gamma_m)$.
(ii) For a fixed $\phi \in H_{00}^{-1/2}(\Gamma_m)$, the set of data T for which there exists a solution $u \in H^1(\Omega)$ to the Cauchy problem (1) is everywhere dense in $H_{00}^{1/2}(\Gamma_m)$.

Two proofs of this theorem, based on Hahn-Banach theorem and penalty method, are given in [16].

Following [11], we introduce now two distinct fields u_1 and u_2 solution of well-posed problems which differ by their boundary conditions. We attribute to each of them one data on Γ_m and one unknown on Γ_u. Then, we have :

$$\begin{cases} -\nabla \cdot \big(k(x)\nabla u_1\big) = f \text{ in } \Omega \\ u_1 = T \text{ on } \Gamma_m \\ k(x)\nabla u_1 \cdot n = \eta \text{ on } \Gamma_u \end{cases} \tag{3}$$

$$\begin{cases} -\nabla \cdot \big(k(x)\nabla u_2\big) = f \text{ in } \Omega \\ u_2 = \tau \text{ on } \Gamma_u \\ k(x)\nabla u_2 \cdot n = \phi \text{ on } \Gamma_m \end{cases} \tag{4}$$

We denote $a_i(\cdot,\cdot)$ and $l_i(\cdot)$, $i = 1, 2$ the bilinear and linear forms associated to the weak forms of the problems (3) and (4) respectively. They are given by :

$$a_i(\tilde{u}_i, v) = \int_\Omega k(x)\nabla\tilde{u}_i \nabla v \; dx \quad i = 1, 2 \tag{5}$$

$$l_1(v) = \int_\Omega fv \; dx - a_1(\bar{u}_1, v) + \langle \eta, v \rangle_{1/2,00,\Gamma_u} \tag{6}$$

$$l_2(v) = \int_\Omega fv \; dx - a_2(\bar{u}_2, v) + \langle \phi, v \rangle_{1/2,00,\Gamma_m} \tag{7}$$

where $\bar{u}_1$ and $\bar{u}_2$ are the lifting of the boundary conditions (T, η) and (τ, ϕ) respectively and $\tilde{u}_i = u_i - \bar{u}_i$, $i = 1, 2$. Then, we have by summation the following weak problem :

$$\begin{aligned} \text{Find} \quad & u = (\tilde{u}_1, \tilde{u}_2) \in V \text{ such that} \\ & a(u, v) = L(v),\ \forall\ v = (v_1, v_2) \in V \\ \text{with} \quad & a(u, v) = a_1(\tilde{u}_1, v_1) + a_2(\tilde{u}_2, v_2) \\ \text{and} \quad & L(v) = l_1(v_1) + l_2(v_2) \end{aligned} \tag{8}$$

where $V = H^1_{0,\Gamma_m}(\Omega) \times H^1_{0,\Gamma_u}(\Omega)$ and $\|v\|_V = (\|v_1\|^2_{1,\Omega} + \|v_2\|^2_{1,\Omega})^{1/2}$ is the norm associated to the space V. It is easy to show that the linear form $L(\cdot)$ is continuous and that the bilinear form $a(\cdot,\cdot)$ is continuous and V-elliptic. Then, by the Lax-Milgram theorem, the weak problem (8) admits a unique solution.

We consider now the following energy-like functional in order to compare the fields u_1 and u_2 :

$$E(\eta,\tau)=\frac{1}{2}\int_\Omega k(x)\big(\nabla u_1(\eta)-\nabla u_2(\tau)\big)^2\,dx \tag{9}$$

and the following minimization problem :

$$\begin{cases}(\varphi,t)=\underset{(\eta,\tau)\in\mathscr{U}}{\operatorname{argmin}}\,E(\eta,\tau), \quad \mathscr{U}=H_{00}^{-1/2}(\Gamma_u)\times H_{00}^{1/2}(\Gamma_u)\\ \text{with } u_1 \text{ and } u_2 \text{ solutions of (3) and (4) respectively.}\end{cases} \tag{10}$$

Using the convexity of the space $\mathscr{U}$, and the existence and uniqueness of the Cauchy problem solution in the case of compatible data, we are able to prove that the solution (η^*,τ^*) of the minimization problem (10), if it exists and is unique, is solution of the data completion problem up to an arbitrary additive constant for the Dirichlet unknown τ. In other words, if $(\eta_d,\tau_d)\in\mathscr{U}$ is solution of the data completion problem, $\eta^*=\eta_d,\ \tau^*=\tau_d+\kappa,$ where κ is a constant.

Remark 2 :

(i) When $E(\eta,\tau)$ reaches its minimum, $\nabla u_1(\eta^*)=\nabla u_2(\tau^*)$.

(ii) The energy-like functional is quadratic.

Remark 3 : The minimization problem (10) can be re-formulated as an optimal control problem as defined in [17]. We assume more regularity on the field u_2, say $u_2\in H^2(\Omega)\cap H^1_{0,\Gamma_m}(\Omega)$. Thereby, we define the operators $A\in\mathscr{L}(V',V)$ and $B\in\mathscr{L}(\mathscr{U},V')$ given by :

$$(Au,v)_{V',V}=a(u,v) \quad\text{and}\quad \big(B(\eta,\tau),v\big)_{V',V}=\langle\eta,v\rangle_{1/2,00,\Gamma_u}-\langle k(x)\nabla v\cdot n,\tau\rangle_{1/2,00,\Gamma_u} \tag{11}$$

and $F\in V'$ such that $(F,v)_{V',V}=(f,v)_{L^2(\Omega)}$. Then, the weak problem (8) can be written as

$$Au(\eta,\tau)=F+B(\eta,\tau). \tag{12}$$

Furthermore, we define an operator $C\in\mathscr{L}\big(V,L^2(\Omega)\big)$ such that the energy-like functional could be written as follows :

$$E(\eta,\tau)=\|Cu(\eta,\tau)\|_{L^2(\Omega)} \tag{13}$$

The functional being convex, it can be proven that, if it exists, the optimal control is unique. However, it is well-known that conditions that guarantee existence can be hardly described. Nevertheless, it appears not as restrictive. Indeed, as seen later, even without this condition one can produce a stable algorithm for finding a numerical solution (see [14]).

This formulation enables us to characterize the optimal control. We introduce the adjoint state $v=(v_1,v_2)\in V$. If (η,τ) is the optimal control, v_1 and v_2 are solution of the two following adjoint problems :

$$\begin{cases}\nabla\cdot\big(k(x)\nabla v_1\big)=0 \text{ in } \Omega\\ v_1=0 \text{ on } \Gamma_m\\ k(x)\nabla v_1\cdot n=\eta-k(x)\nabla u_2\cdot n \text{ on } \Gamma_u\end{cases} \tag{14}$$

$$\begin{cases}\nabla\cdot\big(k(x)\nabla v_2\big)=0 \text{ in } \Omega\\ v_2=0 \text{ on } \Gamma_u\\ k(x)\nabla v_2\cdot n=\phi-k(x)\nabla u_1\cdot n \text{ on } \Gamma_m\end{cases} \tag{15}$$

The gradient of the related functional is then given by :

$$\nabla E(\eta,\tau)=\big(v_{1|\Gamma_u},-k(x)\nabla v_2\cdot n_{|\Gamma_u}\big) \tag{16}$$

This optimal control problem is equivalent to a constrained optimization problem (see [18]) by introducing the following Lagrangian :

$$\mathscr{L}(\eta,\tau,u,v)=E(\eta,\tau)-\langle v,Au-F-B(\eta,\tau)\rangle_{V',V} \tag{17}$$

4 Finite element discretization and error estimation

4.1 Finite element discretization

Let X_h be the finite element space for which the following classical assumptions are verified :

(i) Ω is polyhedral domain in $\mathbb{R}^d, d = 2, 3$.

(ii) $\mathscr{T}_h$ is a regular triangulation of $\bar{\Omega}$ i.e. $h = \max_{K \in \mathscr{T}_h} h_K \to 0$ and $\max_{K \in \mathscr{T}_h} \dfrac{h_K}{\rho_K} \leq c$ with c independent constant on h, h_K the element K diameter and ρ_K the K inscribed circle diameter.

(iii) Γ_u and Γ_m can be written exactly as a union of faces of some finite elements $K \in \mathscr{T}_h$.

(iv) The family $(K, P_K, \Sigma_K), K \in \mathscr{T}_h$ for all h is affine-equivalent to a unique reference finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ of class $\mathscr{C}^0$.

(v) The following inclusion is satisfied : $P_l(\hat{K}) \subset \hat{P} \subset H^1(\hat{K})$ for $l \geq 1$.

These assumptions imply that $X_h \subset H^1(\Omega)$. We define the following spaces :

$$X_{uh} = \{v_h \in X_h; v_{h|\Gamma_u} = 0\}$$
$$X_{mh} = \{v_h \in X_h; v_{h|\Gamma_m} = 0\}$$

and $V_h = X_{mh} \times X_{uh} \subset V$ the finite dimensional approximation space. So, we have the discrete problem associated to the weak problem (8) :

$$\begin{array}{ll} \text{Find} & u_h \in V_h \text{ such that} \\ & a(u_h, v_h) = L(v_h),\ \forall\ v_h \in V_h \end{array} \tag{18}$$

The Lax-Milgram theorem guarantees that (18) admits a unique solution.

4.2 Convergence analysis

Using standard procedures (see [19], Theorem 3.2.2.), we report the following error estimate :

Proposition 4.1 In addition to the assumptions stated above, assume that there exists an integer $l \geq 1$ *such that the following inclusion is satisfied :*

$$H^{l+1}(\hat{K}) \subset \mathscr{C}^s(\hat{K}) \text{ with continuous injection} \tag{19}$$

where s is the maximal order of partial derivatives occurring in the definition of the set $\hat{\Sigma}$.

Then, if the solution $u \in V$ of the variational problem (8) is also in the space $\left(H^{l+1}(\hat{K})\right)^2$, there exists a constant C independent on h such that

$$\|u - u_h\|_V \leq Ch^l(|u_1|^2_{l+1,\Omega} + |u_2|^2_{l+1,\Omega})^{1/2} \tag{20}$$

where $u_h \in V_h$ is the discrete solution.

5 Noisy data, apriori error estimates and stopping criteria

5.1 A priori error estimates and data noise effects

In the case of given perturbed data, say (ϕ^δ, T^δ), problem (18) writes as:

$$\begin{array}{ll} \text{Find} & u_h^\delta = (u_{1h}^\delta, u_{2h}^\delta) \in V_h \text{ such that} \\ & a(u_h^\delta, v_h) = L^\delta(v_h),\ \forall\ v_h \in V_h \\ \text{where} & L^\delta(\cdot) \text{ is the linear form with noisy data } (\phi^\delta, T^\delta) \end{array} \tag{21}$$

Proposition 5.1 Under assumptions of proposition 4.1, if the solution $u \in V$ of the variational problem (8) is also in the space $\left(H^{l+1}(\hat{K})\right)^2$, then there exist two constants C_1 and C_2 independent on h and data such that

$$\|u - u_h^\delta\|_V \le C_1 h^l (|u_1|^2_{l+1,\Omega} + |u_2|^2_{l+1,\Omega})^{1/2} + C_2(\|T - T^\delta\|^2_{1/2,00,\Gamma_m} + \|\phi - \phi^\delta\|^2_{-1/2,00,\Gamma_m})^{1/2} \quad (22)$$

where u_h^δ is the solution of the discrete problem (21) associated to the noisy Cauchy problem.

Proof Using the V-ellipticity property of the bilinear form $a(\cdot,\cdot)$, we have

$$\alpha \|u_h - u_h^\delta\|_V^2 \le L(u_h - u_h^\delta) - L^\delta(u_h - u_h^\delta) \quad (23)$$

According to the trace theorem (see [20]), the trace operator is continuous and there exist two lifting operators $\mathscr{R}_1 : H_{00}^{1/2}(\Gamma_m) \to H^1(\Omega)$ and $\mathscr{R}_2 : H_{00}^{-1/2}(\Gamma_m) \to H^1(\Omega)$ which are continuous and linear. Then, there exists a constant $C > 0$ such that

$$\alpha \|u_h - u_h^\delta\|_V^2 \le C \|u_h - u_h^\delta\|_V \left(\|\mathscr{R}_1(T - T^\delta)\|^2_{1,\Omega} + \|\mathscr{R}_2(\phi - \phi^\delta)\|^2_{1,\Omega}\right)^{1/2} \quad (24)$$

and by continuity, there exist two constants $M_1,\ M_2$ such that

$$\|u_h - u_h^\delta\|_V \le \frac{C}{\alpha}\left(M_1 \|T - T^\delta\|^2_{1/2,00,\Gamma_m} + M_2 \|\phi - \phi^\delta\|^2_{-1/2,00,\Gamma_m}\right)^{1/2} \quad (25)$$

Using the triangular inequality, we have

$$\|u - u_h^\delta\|_V = \|u - u_h + u_h - u_h^\delta\|_V \le \|u - u_h\|_V + \|u_h - u_h^\delta\|_V \quad (26)$$

We obtain (22) by applying (20) and (25) on (26).

5.2 Stopping criteria for the minimization process

When noise is introduced on the Cauchy data, we observe during the optimization process that the error reaches a minimum before increasing very fast and leading to a numerical explosion. At the same time, the energy-like functional attains asymptotically a minimal threshold, which is a strictly positive constant depending on the noise. Notice that this constant vanishes for compatible Cauchy data. Now, the aim is to theoretically determine this threshold in order to propose a stopping criteria depending on the noise rate. This criteria will allow to stop the minimization process just before numerical explosion. Let

$$E_h^\delta(\eta, \tau) = \frac{1}{2}\int_\Omega k(x)\left(\nabla u_{1h}^\delta(\eta) - \nabla u_{2h}^\delta(\tau)\right)^2 dx \quad (27)$$

be the perturbed discrete functional.

Proposition 5.2 Under assumptions of proposition 4.1, if the solution $u \in V$ of the variational problem (18) is also in the space $\left(H^{l+1}(\hat{K})\right)^2$ and if (η^*, τ^*) is the solution of the minimization problem (10), then there exist two constants C_1 and C_2 independent on h and data such that

$$E_h^\delta(\eta^*, \tau^*) \le C_1 h^{2l}(|u_1|^2_{l+1,\Omega} + |u_2|^2_{l+1,\Omega}) + C_2(\|T - T^\delta\|^2_{1/2,00,\Gamma_m} + \|\phi - \phi^\delta\|^2_{-1/2,00,\Gamma_m}) \quad (28)$$

Proof Let (η^*, τ^*) be the solution of the minimization problem (10) with compatible Cauchy data. After some algebraic operations and taking into account the fact that $\nabla u_1(\eta^*) = \nabla u_2(\tau^*)$, we can write :

$$E_h^\delta(\eta^*, \tau^*) - E(\eta^*, \tau^*) = \frac{1}{2}\int_\Omega k(x)\left[\left(\nabla u_{1h}^\delta(\eta^*) - \nabla u_1(\eta^*)\right) - \left(\nabla u_{2h}^\delta(\tau^*) - \nabla u_2(\tau^*)\right)\right]^2 dx \quad (29)$$

As seen previously $E(\eta^*, \tau^*) = 0$. Consequently :

$$E_h^\delta(\eta^*, \tau^*) \le \|k\|_{L^\infty(\Omega)}\left(|\nabla u_{1h}^\delta(\eta^*) - \nabla u_1(\eta^*)|^2_{1,\Omega} + |\nabla u_{2h}^\delta(\tau^*) - \nabla u_2(\tau^*)|^2_{1,\Omega}\right) \quad (30)$$

and then

$$E_h^\delta(\eta^*,\tau^*) \leq \|k\|_{L^\infty(\Omega)}\|u-u_h^\delta\|_V^2 \tag{31}$$

Therefore, using proposition 5.1, we derive (28).

We immediately conclude that, when the discrete functional with noisy data (27) reaches its minimum, for h sufficiently small, we have by (28) :

$$E_h^\delta(\eta^*,\tau^*) \sim O\left(\|T-T^\delta\|^2_{1/2,00,\Gamma_m} + \|\phi-\phi^\delta\|^2_{-1/2,00,\Gamma_m}\right) \tag{32}$$

In order to propose stopping criteria based on these theoretical estimates, let us denote by (X_η^j, X_τ^j) the discrete optimization variables related to the unknown boundary conditions (η,τ) where j points out on the current iteration. We denote by $E_j(X_\eta^j, X_\tau^j)$ the value of the discrete noisy functional $E_h^\delta(\eta,\tau)$ at the iteration j. For more readability, we write $E_j := E_j(X_\eta^j, X_\tau^j)$.

A consistent stopping criteria, based on the described behavior of $E_h^\delta(\cdot,\cdot)$ and the estimate (28), could be :

$$|E_j - E_{j-1}| \leq (\|T-T^\delta\|^2_{1/2,00,\Gamma_m} + \|\phi-\phi^\delta\|^2_{-1/2,00,\Gamma_m}) \tag{33}$$

6 Numerical issues

6.1 Numerical procedure

Let us describe the calculation method of the required elements for the optimization procedure, specifically the adjoint states and the gradient of the functional. Assume that the triangulation $\mathscr{T}_h$ of Ω is characterized by n nodes. Let p and q denote the number of nodes on the boundaries Γ_u and Γ_m respectively and $(\omega_i)_{1\leq i\leq n} = (\omega_{1i},\omega_{2i})_{1\leq i\leq n}$ the canonical basis of V_h. We write X_η and X_τ as the unknowns. The vectors U_1 and U_2 correspond to the fields u_1 and u_2. We introduce the following notations, $(K_1)_{kl} = a_1(\omega_{1k},\omega_{1l})$, $(K_2)_{kl} = a_2(\omega_{2k},\omega_{2l})$, $(F_1)_k = l_1(\omega_{1k})$, $(F_2)_k = l_2(\omega_{2k})$. The bilinear forms being similar, we note $K = K_1 = K_2$.

Following [12, 13], we have the linear systems :

$$\begin{cases} KU_1 + L_m^T p_1 = F_1(X_\eta) \\ L_m U_1 = \mathrm{T}^\delta \end{cases} \tag{34}$$

$$\begin{cases} KU_2 + L_u^T p_2 = F_2(\Phi^\delta) \\ L_u U_2 = X_\tau \end{cases} \tag{35}$$

where $L_u \in \mathscr{M}_{p\times n}(\mathbb{R})$ and $L_m \in \mathscr{M}_{q\times n}(\mathbb{R})$ are matrices that contain only 0 and 1, p_1 and p_2 are Lagrange multipliers laying down Dirichlet conditions.

Based on (9) and (17), we can write the discrete functional :

$$E(X_\eta,X_\tau) = \frac{1}{2}(U_1-U_2)^T K (U_1-U_2) \tag{36}$$

and the discrete lagrangian :

$$\begin{aligned}\mathscr{L}(U_1,U_2,\lambda_1,\lambda_2;X_\eta,X_\tau) = E(X_\eta,X_\tau) &- \begin{bmatrix}\lambda_1\\ q_1\end{bmatrix}^T \begin{bmatrix} KU_1 + L_m^T p_1 - F_1 \\ L_m U_1 - \mathrm{T}^\delta\end{bmatrix} \\ &- \begin{bmatrix}\lambda_2\\ q_2\end{bmatrix}^T \begin{bmatrix} KU_2 + L_u^T p_2 - F_2 \\ L_u U_2 - X_\tau\end{bmatrix}\end{aligned} \tag{37}$$

Let (X_η^*, X_τ^*) be the otpimum. Derivating this lagrangian, we have :

$$\frac{\partial\mathscr{L}}{\partial X_\eta} = \frac{dU_1}{dX_\eta}\big(2K(U_1-U_2) - \lambda_1^T K - q_1^T L_m\big) - \lambda_1^T L_m^T \frac{dp_1}{dX_\eta} - \lambda_1^T \frac{dF_1}{dX_\eta} \tag{38}$$

$$\frac{\partial\mathscr{L}}{\partial X_\tau} = \frac{dU_2}{dX_\tau}\big(2K(U_2-U_1) - \lambda_2^T K - q_2^T L_u\big) - \lambda_2^T L_u^T \frac{dp_2}{dX_\tau} + L_u L_u^T q_2 \tag{39}$$

Therefore, given that $\frac{\partial \mathscr{L}}{\partial X_\eta}(X_\eta^*) = 0$ and $\frac{\partial \mathscr{L}}{\partial X_\tau}(X_\tau^*) = 0$ and the adjoint states corresponding to the Lagrange multipliers (see (17)), we have by identification the discrete adjoint problems :

$$\left\{ \begin{array}{l} K\lambda_1 + L_m^T q_1 = K(U_1 - U_2) \\ L_m \lambda_1 = 0 \end{array} \right. \quad (40) \qquad \left\{ \begin{array}{l} K\lambda_2 + L_u^T q_2 = K(U_2 - U_1) \\ L_u \lambda_2 = 0 \end{array} \right. \quad (41)$$

and the gradient of the discrete functional is then given by :

$$\nabla E(X_\eta, X_\tau) = \left[\begin{array}{l} L_u \lambda_1 \\ L_u[K(U_2 - U_1) - K\lambda_2] \end{array} \right] \tag{42}$$

We consider here the case of real applications where we have only measured and noisy data $(\mathrm{T}^\delta, \Phi^\delta)$ given with a noise rate $0 < a < 1$. We are then not able to calculate exactly the norm of the difference between the exact and noisy data which are involved in the stopping criteria (33). We have therefore to estimate these norm. We have :

$$\mathrm{T}(x) - a\mathrm{T}(x) \le \mathrm{T}^\delta(x) \le \mathrm{T}(x) + a\mathrm{T}(x), \quad \forall\, x \in \Gamma_m \tag{43}$$

$$\iff \frac{-a}{1-a}\mathrm{T}^\delta(x) \le \mathrm{T}(x) - \mathrm{T}^\delta(x) \le \frac{a}{1+a}\mathrm{T}^\delta(x) \tag{44}$$

$$\text{and then} \quad \|\mathrm{T} - \mathrm{T}^\delta\|^2_{1/2,00,\Gamma_m} \le \max\left\{\frac{a}{1-a}, \frac{a}{1+a}\right\} \|\mathrm{T}^\delta\|_{1/2,00,\Gamma_m} \tag{45}$$

Proceeding by the same way for the Neumann boundary condition, the stopping criteria (33) can be written as follows :

$$|E_j - E_{j-1}| \le \frac{a^2}{(1-a)^2}\left(\|\mathrm{T}^\delta\|^2_{1/2,00,\Gamma_m} + \|\Phi^\delta\|^2_{-1/2,00,\Gamma_m}\right) \tag{46}$$

6.2 Numerical results

We consider the following Cauchy problem on the domain Ω given by figure (2) :

$$\left\{ \begin{array}{l} \triangle u = 0 \text{ in } \Omega \\ u = f_D \text{ on } \Gamma_m \\ \frac{\partial u}{\partial n} = f_N \text{ on } \Gamma_m \end{array} \right. \tag{47}$$

where f_D and f_N are the Cauchy data extracted from the exact solution which we intend to approximate.

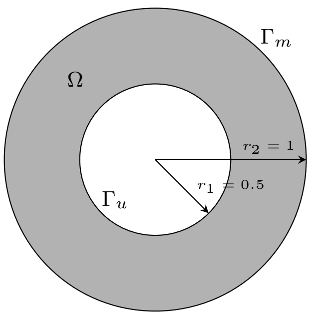
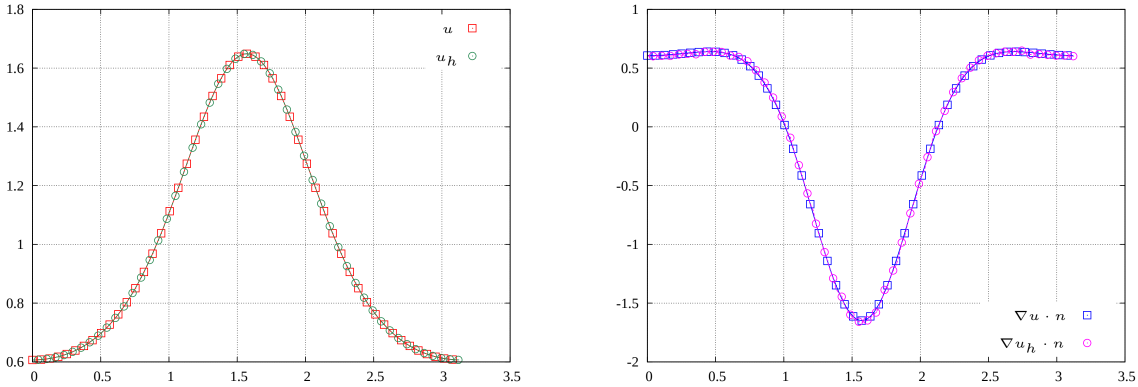


Figure 2: Ring

6.2.1 Analytic example

The figure 3 represents the exact analytic solution $u(x, y) = e^x \cos(y)$ and the finite element solution of the data completion problem obtained by energy-like functional minimization. We can see that the recovered temperature and heat flux are close to the exact ones.



(a) Dirichlet condition on Γ_u (b) Neumann condition on Γ_u

Figure 3: Exact ($\square$) and identified ($\circ$) boundary conditions, $h = 0.03$

The figure 6 represents the finite element discretization error with respect to the maximum edge size of the mesh. This result is in agreement with the theoretical error estimates (20).

We introduce a gaussian random noise on data with an amplitude which depends on a rate a. The figures 4 and 5 represent the error and the energy-like functional at each iteration for different noise rates. These behaviors make it necessary to introduce criteria to stop the optimization process.

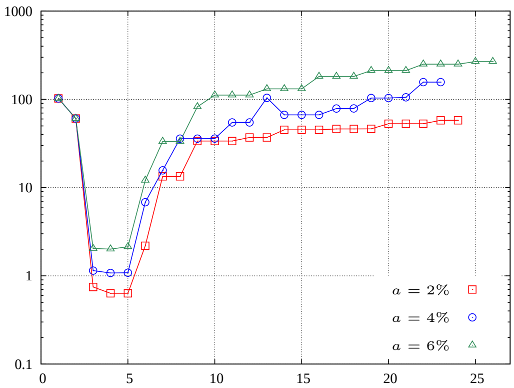


Figure 4: Evolution of $\|u - u_h\|_{1,\Omega}$ during the optimization procedure for different noise rates

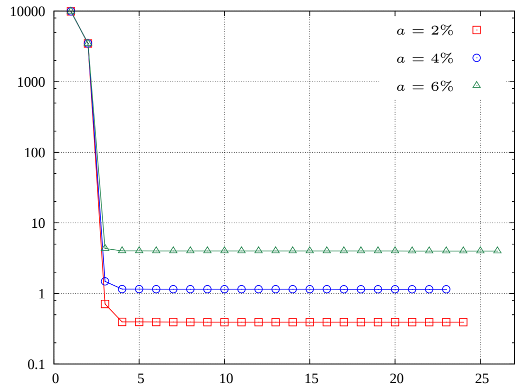


Figure 5: Evolution of $E(\eta, \tau)$ during the optimization procedure for different noise rates

Next we choose h such that the finite elements error could be negligible in comparison with error due to noise and we observe error and functional behaviors with respect to the noise norm. These results, presented in the figure 7, are in agreement with the error estimates (22) and (28).

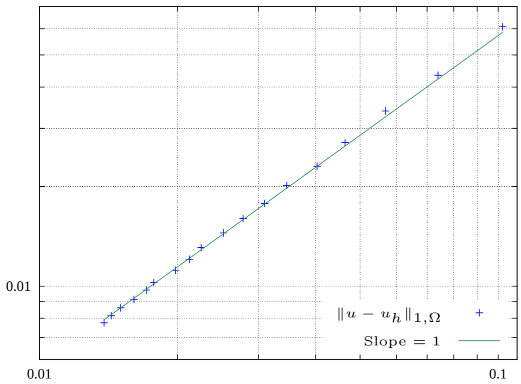


Figure 6: Evolution of $\|u - u_h\|_{1,\Omega}$ with respect to h

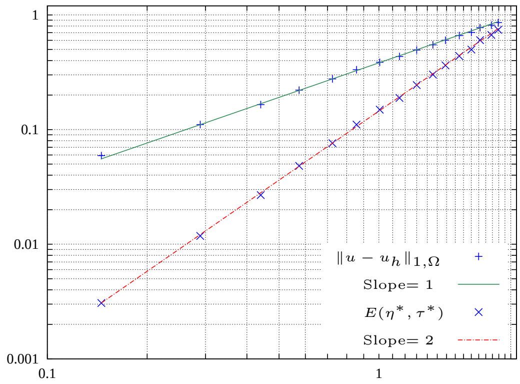
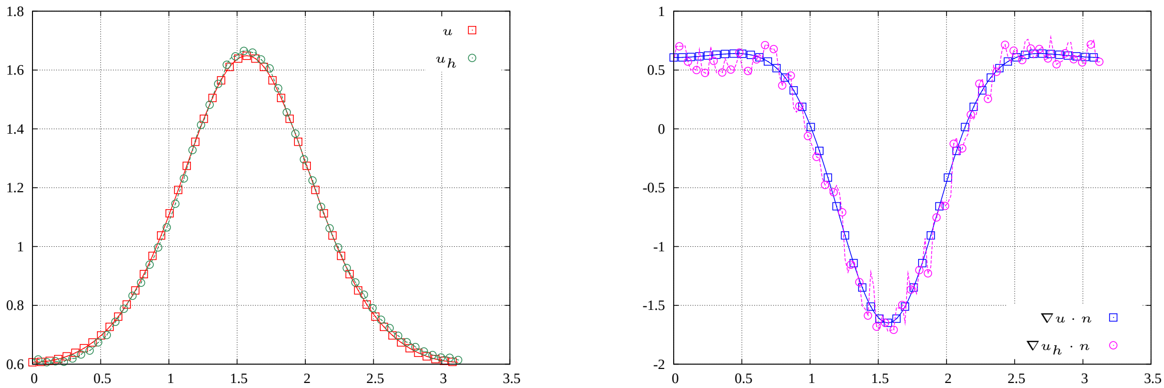


Figure 7: Evolution of $\|u - u_h\|_{1,\Omega}$ and $E(\eta, \tau)$ with respect to the noise norm.

The stopping criteria defined by (33) allows one to identify a consistent solution, as shown in figure 8, otherwise the solution of the optimization algorithm numerically explodes.



(a) Dirichlet condition on Γ_u

(b) Neumann condition on Γ_u

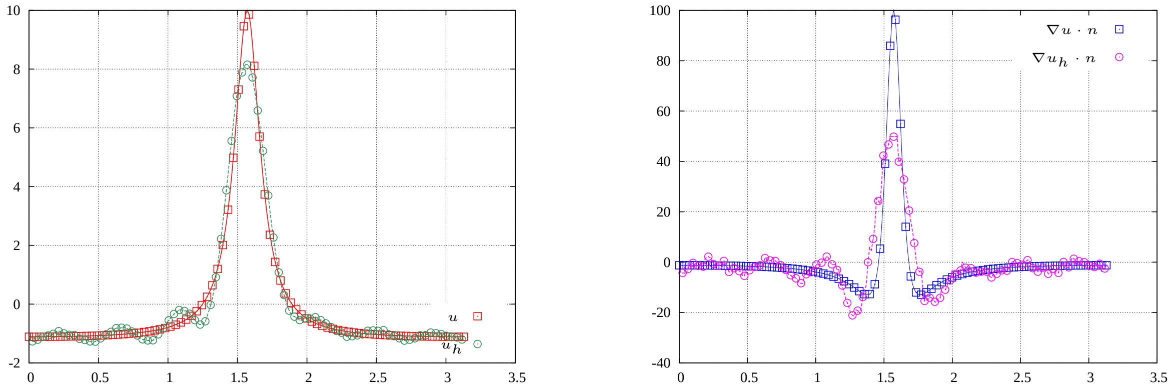
Figure 8: Exact ($\square$) and identified ($\circ$) boundary conditions with noisy data, $a = 4\%$, $h = 0.03$

6.2.2 Source point and stratified inner fluid examples

The next source point example deals with the reconstruction of singular data, coming from

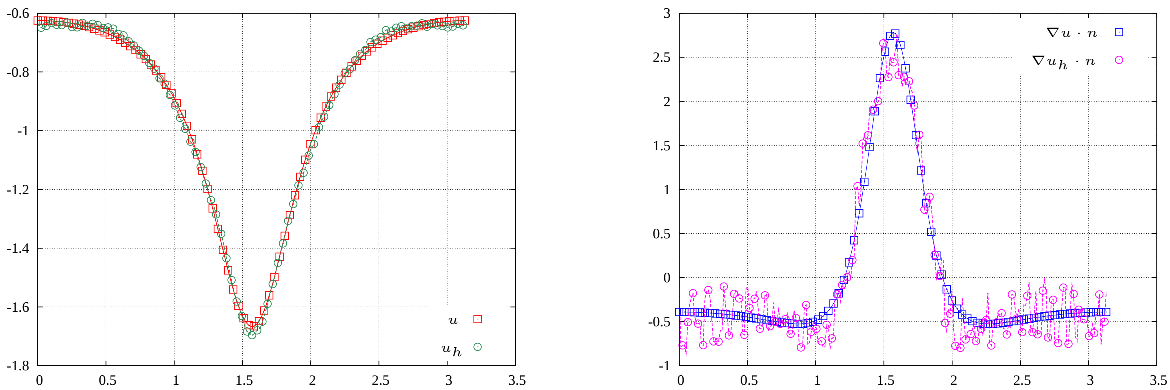
$$u(x, y) = Re(\frac{1}{z - r}), \quad \text{where } z = x + iy \tag{48}$$

where r is the position of the source point on the abscissa axis. Numerical results are illustrated by figure 9 in the case that the source point is in the vicinity of the inner boundary and figure 10 if the source point is in the vicinity of the outer boundary.



(a) Dirichlet condition on Γ_u (b) Neumann condition on Γ_u

Figure 9: Exact ($\square$) and identified ($\bigcirc$) boundary conditions with noisy data, $r = 0.4$, $a = 4\%$, $h = 0.02$



(a) Dirichlet condition on Γ_u (b) Neumann condition on Γ_u

Figure 10: Exact ($\square$) and identified ($\bigcirc$) boundary conditions with noisy data, $r = 1.1$, $a = 4\%$, $h = 0.02$

Now, we explore the efficiency of proposed stopping criteria on the so-called stratified inner fluid case. We consider therefore the reconstruction of temperature and flux in a pipeline of infinite length. This application arises in several industrial processes. Indeed, knowledge of temperature on internal wall of a pipeline is necessary for controlling the material's safety : a stratified inner fluid generates mechanical stresses, which may cause damage such as cracks. We assume that the temperature does not depend on the longitudinal coordinate. We consider then the following problem on the geometry defined by figure 2 :

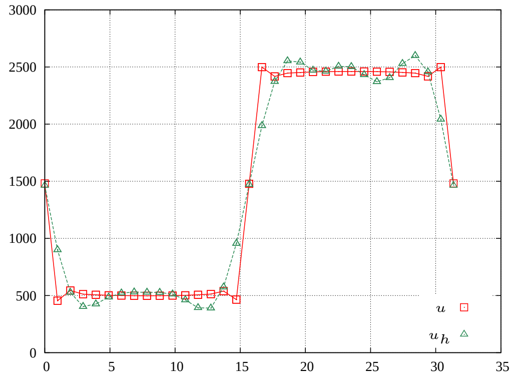
$$\begin{cases} \nabla \cdot (k \nabla u) = 0 \text{ in } \Omega \\ k \nabla u \cdot n + \alpha u = T \text{ on } \Gamma \end{cases} \tag{49}$$

where $k = 17\ W.m^{-1}.^{\circ}\mathrm{C}^{-1}$ is the constant thermal conductivity, T is the temperature, α is the Fourier coefficient, Γ_u is partitioned into two parts, the lower half circle $\Gamma_{u,lo} = \{(x, y) \in \Gamma_u; y < 0\}$ and the upper half circle $\Gamma_{u,up} = \{(x, y) \in \Gamma_u; y \geq 0\}$. The coefficients values are given in table 1.

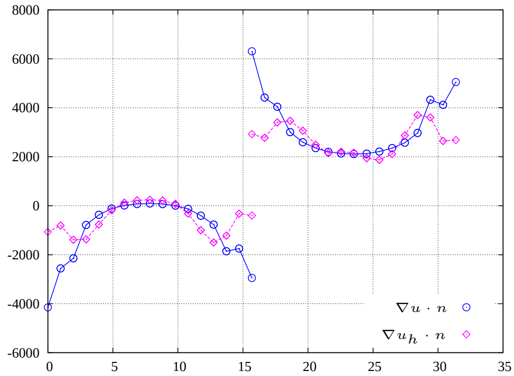
	T (°C)	α ($W.m^{-2}$.°C^{-1})
Γ_m	20	12
$\Gamma_{u,up}$	250	1000
$\Gamma_{u,lo}$	50	1000

Table 1: Coefficients values for stratified inner fluid test

The Cauchy data are generated by solving the forward problem defined by (49). Then, a random noise is applied on Dirichlet data and we assume that the flux is exactly known on Γ_m. The figure 11 shows the recovered temperature and heat flux in comparison to the data given by numerical resolution of (49). Notice that the reconstructed field is close to the solution to be recovered.



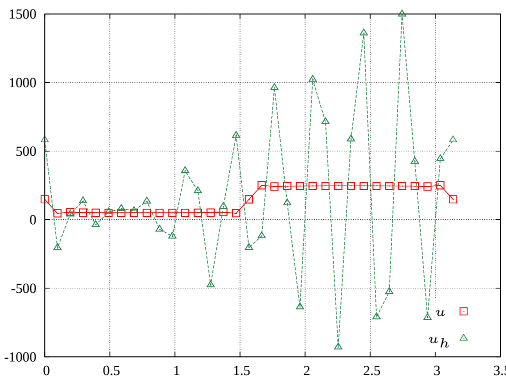
(a) Dirichlet condition on Γ_u



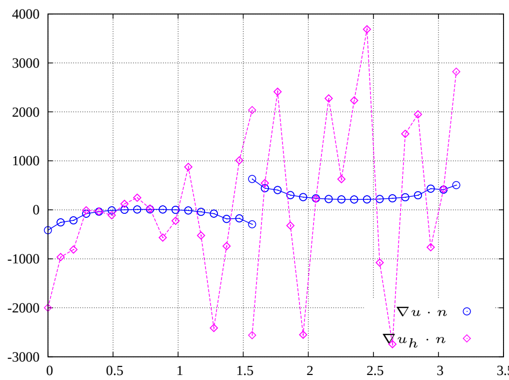
(b) Neumann condition on Γ_u

Figure 11: Exact (□ resp. ○) and Identified (△ resp. ⋄) temperature and flux respectively on Γ_u with noisy data, $a = 4\%$, $h = 0.1$.

Finally, in order to illustrate the efficiency of the given stopping criteria, we increase the noise rate up to 10% and perform numerical experiments. Figure 12 shows the solution of the generic optimization algorithm. However figure 13 shows the solution of the optimization algorithm with the stopping criteria defined by (33). A numerical explosion without the proposed stopping criteria is then clearly observed.

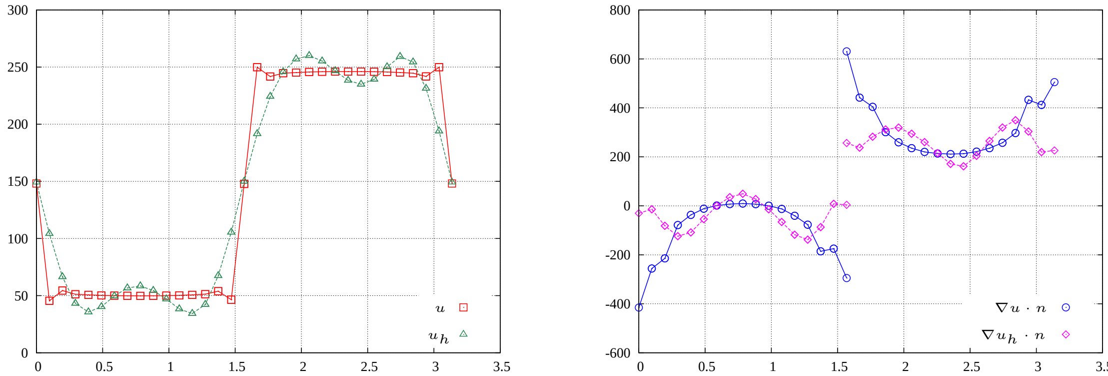


(a) Dirichlet condition on Γ_u



(b) Neumann condition on Γ_u

Figure 12: Exact (□ resp. ○) and Identified (△ resp. ⋄) temperature and flux respectively on Γ_u with noisy data and classical stopping criteria, $a = 10\%$, $h = 0.1$.



(a) Dirichlet condition on Γ_u (b) Neumann condition on Γ_u

Figure 13: Exact (□ resp. ○) and Identified (△ resp. ⋄) temperature and flux respectively on Γ_u with noisy data, $a = 10\%$, $h = 0.1$.

7 Conclusion

In this work, we stated the Cauchy problem as a minimization one and presented classical theoretical results. Then, we gave the finite element discretization and performed convergence analysis. We derived a priori error estimates taking into account noisy data. Then we proposed stopping criteria depending on the noise rate in order to control the numerical instability of the minimization process due to noisy data. We proposed a numerical procedure and performed numerical experiments in agreement with error estimates. We illustrated the robustness and efficiency of the proposed stopping criteria, especially in the case of singular data. The numerical analysis of noise effects and derivation of stopping minimization criteria for parabolic (see [21, 22]) and hyperbolic problems is under consideration. It will be a subject of a forthcoming paper.

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